Mutually entangled quantum Latin squares

Simeon Ball and **Robin Simoens**

Latin squares

LS(n): matrix $A \in \{1,\dots,n\}^{n\times n}$ such that each row and column contains n different elements.

1		2		
				1
3				
	4			2
		3	2	

Quantum Latin squares

QLS(n): matrix $A \in (\mathbb{C}^n)^{n\times n}$ such that each row and column is an orthonormal basis of $\mathbb{C}^n$.

$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$
$\vert 2\rangle$	$\vert 3\rangle$	$\vert 1\rangle$
$\vert 3\rangle$	$\vert 1\rangle$	$\vert 2\rangle$

$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$	$\vert 4\rangle$
$\vert 2\rangle$	$\vert 1\rangle$	$\vert 4\rangle$	$\vert 3\rangle$
$\vert 3\rangle$	$\vert 4\rangle$	$\vert +\rangle$	$\vert -\rangle$
$\vert 4\rangle$	$\vert 3\rangle$	$\vert -\rangle$	$\vert +\rangle$

$|i\rangle$ denotes the ith standard basis vector and $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$

Why care?

Motivation from:

- quantum teleportation
- dense coding
- complex Hadamard matrices
- quantum graph isomorphism
- mutually unbiased bases
- AME states
- perfect tensors

Mutually orthogonal Latin squares

t **MOLS**(n): set $\{A^1,\dots,A^t\}$ of **LS**(n) such that

$$k_1 \neq k_2 \implies \left\{(a_{ij}^{k_1}, a_{ij}^{k_2}) : i,j \in \{1,\dots,n\}\right\} = \{1,\dots,n\}^2.$$

1	2			
		2		
3				2
	4			

		1		
		2		
		3		4
2	5			
	3			

A pair of MOLS is a *Graeco-Latin square*.

What is the best notion of orthogonality in the quantum setting?

Mutually orthogonal quantum Latin squares

t **MOQLS**(n): set $\{A^1,\dots,A^t\}$ of **QLS**(n) such that

$$k_1 \neq k_2 \implies \left\{a_{ij}^{k_1} \otimes a_{ij}^{k_2} : i,j \in \{1,\dots,n\}\right\} \text{ is a basis of } \mathbb{C}^n \otimes \mathbb{C}^n.$$

$\vert -\rangle$	$\vert 3\rangle$	$\vert +\rangle$
$\vert +\rangle$	$\vert -\rangle$	$\vert 3\rangle$
$\vert 3\rangle$	$\vert +\rangle$	$\vert -\rangle$

$\vert 1\rangle$	$\vert 3\rangle$	$\vert 2\rangle$
$\vert 3\rangle$	$\vert 2\rangle$	$\vert 1\rangle$
$\vert 2\rangle$	$\vert 1\rangle$	$\vert 3\rangle$

Mutually entangled quantum Latin squares

t **MEQLS**(n): matrix $A \in \Big(\underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{t}\Big)^{n\times n}$ such that

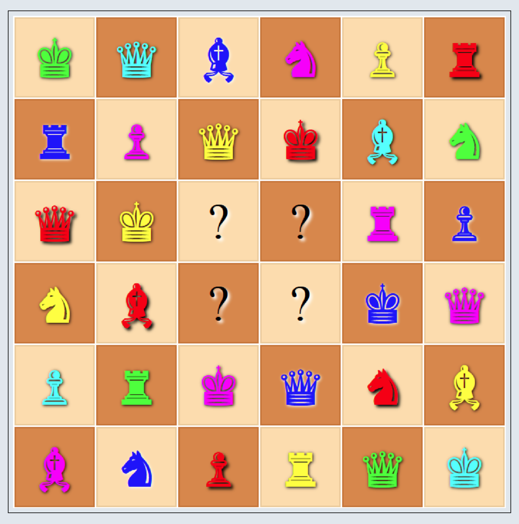
$\frac{1}{n}\sum_{ij} |i\rangle \otimes |j\rangle \otimes a_{ij}$ is *maximally mixed* over all subsystems of size two.

$\vert 1\rangle\otimes\vert 1\rangle$	$\vert 2\rangle\otimes\vert 2\rangle$	$\vert 3\rangle\otimes\vert 3\rangle$
$\vert 2\rangle\otimes\vert 3\rangle$	$\vert 3\rangle\otimes\vert 1\rangle$	$\vert 1\rangle\otimes\vert 2\rangle$
$\vert 3\rangle\otimes\vert 2\rangle$	$\vert 1\rangle\otimes\vert 3\rangle$	$\vert 2\rangle\otimes\vert 1\rangle$

MOLS $\subsetneq$ **MOQLS** $\subsetneq$ **MEQLS**

Thirty-six officers

Question (Euler, 1782). Can six army regiments, each with six officers of different ranks, be arranged in a 6×6 square such that no row or column repeats a rank or regiment?



$\exists$ **2 MOLS(6)**?

Answer (Tarry, 1900). No.

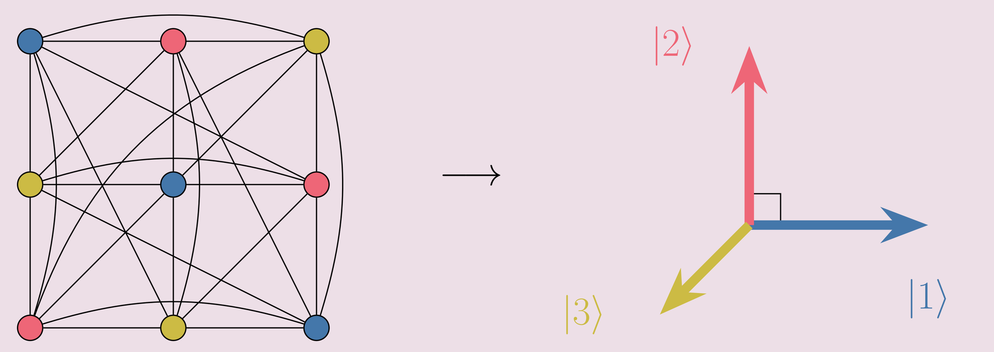
Quantum officers

Theorem (Ball and Simoens, 2026).

$$\nexists \text{ 2 MOQLS}(6)$$

Proof.

- One of **2 MOQLS(6)** is **classical**
- If a **LS**(n) can be extended to **2 MOQLS**(n), its *Latin square graph* has an *orthonormal representation* in $\mathbb{C}^n$



- *LS graph* of **LS(6)** has no *orthonormal representation* in $\mathbb{C}^6$ □

Entangled officers

Theorem (Rather, Burchardt, Bruzda, Rajchel-Mieldzioć, Lakshminarayan and Życzkowski, 2022).

$\exists$ **2 MEQLS(6)**

Link with coding theory

Theorem. t **MOLS**$(n) \iff (t+2, n^2, t+1)_n$ code.

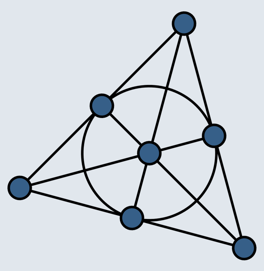
Link with quantum coding theory

Theorem (Ball and Simoens, 2026).
t **MOQLS**$(n) \iff ((t+2, 1, \geq 3))_n$ quantum code with *product structure*.

Link with quantum coding theory

Theorem (Ball and Simoens, 2026).
t **MEQLS**$(n) \iff ((t+2, 1, \geq 3))_n$ quantum code.

Link with finite geometry



Theorem. Let $n \geq 2$.

- There are at most $n-1$ **MOLS**(n).
- $n-1$ **MOLS**$(n) \iff$ projective plane of order n.

Rigidity

Theorem (Musto and Vicary, 2019, Ball and Simoens, 2026). Let $n \geq 2$.

- There are at most $n-1$ **MOQLS**(n).
- $n-1$ **MOQLS**(n) are classical.

Abundance

Theorem (Ball and Simoens, 2026). $\exists$ t **MEQLS**$(n) \iff (t,n) \neq (2,2)$.

$\frac{1}{\sqrt{2}}(\vert 0000\rangle + \vert 1110\rangle)$	$\frac{1}{\sqrt{2}}(\vert 0101\rangle + \vert 1011\rangle)$
$\frac{1}{\sqrt{2}}(\vert 0011\rangle + \vert 1101\rangle)$	$\frac{1}{\sqrt{2}}(\vert 0110\rangle + \vert 1000\rangle)$

4 MEQLS(2)

Sudoku Latin squares

SLS(n^2): **LS**(n^2) where each block has n^2 different elements.

							1	
					2			3
			4					
						5		
4		1	6					
		7	1					
	5					2		
				8			4	
	3		9	1				

MOLS + **SLS** → **MOSLS**

Theorem. There are at most $n^2 - n$ **MOSLS**(n^2).

SudoQ

QSLS(n^2): **QLS**(n^2) where each block is an orthonormal basis of $\mathbb{C}^{n^2}$.

	$\vert 1\rangle$		
		$\vert 4\rangle$	
$\vert +\rangle$			
			$\vert 3\rangle$

MOQLS + **QSLS** → **MOQSLS**

Theorem (Herdies, 2026).
There are at most $n^2 - n$ **MOQSLS**(n^2).

Take-home message

- **MEQLS** are equivalent to quantum error-correcting codes
- They exist in abundance
- **MOQLS** behave like classical objects

Conclusion:
Entanglement reveals deeper structure, but trivialises existence.
MOQLS capture the *true combinatorial structure*.

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